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The generic spacing distribution of the two-dimensional harmonic oscillator

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Abstract. The level spacing distribution of the two-dimensional harmonic oscillator is studied. By choosing a suitable average, a stable distribution is obtained. This is then shown to have a fixed form for the generic frequency ratio. A delta function is also found for a dense class of ratio (with Hausdorff dimension $\frac{1}{2}$). Thus the distribution is unstable under perturbation of the ratio. All of this differs from the universal Poissonian statistics associated with integrable systems.

1. Introduction

Studying dynamical systems on the macroscopic scale, i.e. classical level, the linearity or nonlinearity of the system gives rise to integrable and possibly chaotic systems, respectively, with various mixed states between. When the system is studied at the microscopic level quantum mechanics is involved. The main tool of application is Schrödinger's equation, which is inherently linear whether the original system is linear or not. The question then arises as to what happens to the properties of the classical system (such as chaos) in the transition to the quantum scale, and what clues the quantized system give towards the behaviour of the classical system.

For a classically bounded system a discrete set of energy eigenvalues for the quantized system is obtained for which the statistics have been studied extensively. One such statistical measure is the level-spacing distribution. Here the spacings between consecutive levels are first normalized to have an average of unity and then the probability distribution arising from these spacings can be defined.

Certain universality classes of distributions have been observed corresponding to the type of classical system of interest. A classically chaotic system has statistics associated with those of the eigenvalues of certain random matrix classes; see, for example, [1–4]. For a system with time-reversal invariance, the statistics obtained from the spacings between the eigenvalues of the Gaussian orthogonal ensemble of real-symmetric matrices is found, without the invariance, the statistics from the Gaussian unitary ensemble of complex Hermitian matrices is found. For a classically integrable system the Poissonian statistics of random systems are observed. This last class has been derived in Berry and Tabor [5].

For a classically integrable system the spacing distribution can be introduced in the following manner. The physical system will have a Hamiltonian $H(\boldsymbol{p}, \boldsymbol{q})$ dependent upon $2f$ momentum and position coordinates $\boldsymbol{p}$ and $\boldsymbol{q}$, respectively. If the system is integrable then we can parametrize phase space by the action and angle coordinates $\boldsymbol{I}$ and $\boldsymbol{\alpha}$ respectively, which shows the Hamiltonian to be dependent upon the action coordinates $\boldsymbol{I}$ only, and so has an f variable dependency; see, for example, [3, 6]. We are then able to quantize as

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follows. We have the Hamiltonian $H(\boldsymbol{I})$ so putting $\boldsymbol{I} = \hbar \boldsymbol{m}$ gives energy $E_{\boldsymbol{m}} = H(\hbar \boldsymbol{m})$. Then the values $E_{\boldsymbol{m}}$ are quantized energy values whenever $\boldsymbol{m}$ is part of the integer lattice; see [6, 7] for more details.

For systems that will be investigated, components of $\boldsymbol{I}$ are all non-negative, thus we are only concerned with non-negative coordinates of the lattice.

For each value of energy there is a family of surfaces in $\boldsymbol{m}$ space that fill out the space from the origin, as energy increases ($\boldsymbol{m}$ space is the action space $\boldsymbol{I}$ magnified by a factor $\hbar^n$). Thus denoting U as the (hyper)volume enclosed by the surface and the boundaries of the positive 2^f-ant (from here on referred to as the positive quadrant), then the energy E is a monotonic increasing function of U. Then we can label the values of energy and volume at which a lattice point is intersected (with multiplicity where the spectrum is degenerate) as E_i and U_i, respectively, where $E_{i+1} \geqslant E_i$ and $U_{i+1} \geqslant U_i$.

Since we are considering the unit lattice, over an interval of energy ΔE, the number of lattice points moved through by the energy surface will be best estimated as the change in volume ΔU, thus the normalization factor required to give an average spacing of unity will be $\mathrm{d}U/\mathrm{d}E$. Then from the Taylor expansion, for large enough U

$$U_{i+1} - U_i \simeq (E_{i+1} - E_i)\frac{\mathrm{d}U}{\mathrm{d}E} \tag{1}$$

provided that $E(U)$ behaves sensibly (not exponentially divergent for example). The probability distribution is then introduced as follows. First count the number of values $U_i < U$ that give a spacing in the interval $(s, s + \mathrm{d}s)$, and divide this by the total number of U_i values under U, i.e. U. This ratio defines the distribution $P(s, U)$. The level-spacing distribution $P(s)$ is then defined by $\lim_{U\to\infty} P(s, U)$.

Note that for chaotic systems and mixed systems no such parametrization exists. In the mixed KAM systems, for example, although some invariant tori do exist that could possibly be parametrized by f coordinates, in between lie chaotic horseshoe structures that fill out phase space and so defy any such coordinate scheme. Thus the quantization procedure is unworkable, and the resulting theory inapplicable.

Berry and Tabor [5] apply the method of stationary phase to obtain $P(s) = \mathrm{e}^{-s}$. The proof, however, assumes a curved energy surface in phase space, convex from above. For harmonic oscillators the surface is flat and the method breaks down. The following work is largely a continuation of [5] and that of Pandey *et al* [8, 9], in which it has been observed that harmonic oscillators have a degree of level repulsion, and no fixed spacing distribution. The average over energy was also shown to have an oscillatory behaviour.

The problem is equivalent to studying the behaviour of the normalized spacings between successive terms $m\alpha + n$ where m and n are non-negative integers and α is a fixed, real parameter. Thus the problem is one expressible in terms of number theory alone and can be seen as a question independent of physics if so desired.

In section 2 the results of [5, 8, 9] are derived in a different fashion in which the spacings for the two-dimensional harmonic oscillator are found and the level-spacing distribution shown to be oscillatory. In section 3 a suitable average is taken for which a stable limit is shown to exist. Sections 4 and 5 calculate the average for various values of the frequency ratio, including the generic case. Concluding remarks follow.

2. The spacing distribution

The spacings between the levels are now determined exactly, which is then used to investigate the spacing distribution. For the two-dimensional harmonic oscillator (TDHO) with frequencies ω_1, ω_2 the Hamiltonian and the area enclosed by the energy surface (normalized by a factor $\hbar^{-2}$) in the positive quadrant of phase space are (see [7, 10] for example)

$$H(\boldsymbol{I}) = \boldsymbol{\omega} \cdot \boldsymbol{I}$$

giving

$$U(\boldsymbol{m}) = \frac{(\boldsymbol{\omega} \cdot \boldsymbol{m})^2}{2\omega_1\omega_2} \iff m_1\alpha + m_2 = \sqrt{2\alpha U} \tag{2}$$

where $\alpha (= \omega_1/\omega_2)$ is the frequency ratio.

Consider the surface to be initially through a lattice point (m_1, m_2), then increase the area U until another lattice point $(m_1', m_2') = (m_1, m_2) + \boldsymbol{v}$ is reached. If we put the respective areas as U and $U + \Delta U$ then the spacing obtained is simply the difference in area ΔU by (1). The vector $\boldsymbol{v} = (x, y)$ will be referred to as the shift vector (note that x and y have opposite sign). Then the spacing is

$$s = \sqrt{\frac{2U}{\alpha}} x\alpha + y. \tag{3}$$

This does not take into account the possibility of degeneracy among the energy levels. For such a system the frequency ratio α is rational, say p/q. In these cases, the next lattice point that the surface passes through is found from solutions (x, y) to the equation

$$xp + yq = 1.$$

This has of order $\sqrt{2U/pq}$ solutions (x, y), that each give a shift vector pointing to a lattice point, with positive coordinates. Thus for every non-zero spacing, there are of order $\sqrt{2U/pq}$ zero-spacing contributions that arise from the degeneracy. Then as $U \to \infty$ there is an accumulation of zero-spacing contributions to $P(s, U)$, so a delta function at the origin is obtained for the spacing distribution $P(s)$ in the limit $U \to \infty$. Thus unless the distributions for the irrational cases are all delta functions, the distribution is not stable with respect to perturbation of the frequency ratio. We now consider the irrational cases.

From equation (3) the minimization of $|x\alpha - y|$ is required so that (m_1', m_2') is the next point the energy surface passes through, as U is increased. Continued fractions are the ideal tool for doing this.

Irrational values α have an infinite continued fraction, (see Hardy and Wright [10] for more details) where

$$\alpha = \frac{\omega_1}{\omega_2} = [a_0, a_1, a_2, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots}} \tag{4}$$

for positive integers a_i (a_0 may be zero). There are also the convergents

$$\frac{p_n}{q_n} = [a_0, a_1, \ldots, a_n] \tag{5}$$

which tend to α as $n \to \infty$. In order to determine the spacing disribution, the first step is to obtain an analytic expression for the spacings. The following lemma allows the spacings to be determined exactly.

Lemma 1.1. If $x < q_{n+2}$, then the only values x, y such that

$$0 < x\alpha + y \leqslant \begin{cases} -q_n\alpha + p_n & \text{for odd } n \\ q_n\alpha - p_n & \text{for even } n \end{cases}$$

are the values

$$(x, y) = \begin{cases} (-q_n - rq_{n+1}, p_n + rp_{n+1}) & \text{for odd } n \\ (q_n + rq_{n+1}, -p_n - rp_{n+1}) & \text{for even } n \end{cases}$$

for $r \in \{0, 1, 2, \ldots, a_{n+2} - 1\}$, where $|x\alpha + y|_{r+1} < |x\alpha + y|_r$.

Proof. First define

$$x = \mu q_n + \nu q_{n+1} \qquad y = \mu p_n + \nu p_{n+1} \tag{6}$$

which, with the relation (see [10])

$$p_n q_{n+1} - p_{n+1} q_n = \pm 1$$

gives

$$\mu = \pm(y q_{n+1} - x p_{n+1}) \qquad \nu = \pm(y q_n - x p_n)$$

meaning μ and ν are integers. Thus

$$0 < -x\alpha + y \leqslant -q_n\alpha + p_n \qquad \text{for odd } n$$
$$0 < x\alpha - y \leqslant q_n\alpha - p_n \qquad \text{for even } n$$

becomes

$$\begin{aligned} &0 < \mu(-q_n\alpha + p_n) + \nu(-q_{n+1}\alpha + p_{n+1}) \leqslant -q_n\alpha + p_n \qquad \text{for odd } n \\ &0 < \mu(q_n\alpha - p_n) + \nu(q_{n+1}\alpha - p_{n+1}) \leqslant q_n\alpha - p_n \qquad \text{for even } n. \end{aligned} \tag{7}$$

Now if $q'_n = a'_n q_{n-1} + q_{n-2}$, where $a'_n = [a_n, a_{n+1}, \ldots]$, then we have

$$\alpha - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q'_{n+1}}. \tag{8}$$

Then applying this to both right-hand inequalities in (7) gives

$$\frac{\mu - 1}{q'_{n+1}} \leqslant \frac{\nu}{q'_{n+2}}.$$

Then noting that

$$\frac{q'_{n+1}}{q'_{n+2}} = a'_{n+2}$$

we obtain

$$\nu \geqslant a'_{n+2}(\mu - 1).$$

Similarly, in both cases the left-hand inequality is equivalent to

$$0 < \frac{\mu}{q'_{n+1}} - \frac{\nu}{q'_{n+2}} \iff \nu < a'_{n+2}\mu.$$

So whether n is even or odd

$$a'_{n+2}(\mu - 1) \leqslant \nu < a'_{n+2}\mu. \tag{9}$$

Considering (6) with $x > 0$ and the right-hand side of (9) gives the condition $\mu > 0$. Also the left-hand side gives $x \geqslant (\mu - 1)q'_{n+2} + q_n$ but $x < q_{n+2}$ so $\mu < 2$. Thus since μ is an integer $\mu = 1$ giving

$$0 \leqslant \nu < a'_{n+2}$$

but (6) also means $\nu < a_{n+2}$ if $x < q_{n+2}$, so

$$x\alpha + y = \frac{1}{q'_{n+1}} - \frac{\nu}{q'_{n+2}}$$

which decreases with ν, thus completing the proof. □

These two results are then used to determine the spacings. This is described below.

Theorem 1. For a point $\boldsymbol{m} = (m_1, m_2)$ in the positive quadrant of the lattice such that

$$q_n + rq_{n+1} \leqslant m_1 < q_n + (r+1)q_{n+1} \qquad 0 \leqslant m_2 < p_{n+1} \qquad \text{for odd } n$$
$$0 \leqslant m_1 < q_{n+1} \qquad p_n + rp_{n+1} \leqslant m_2 < p_n + (r+1)p_{n+1} \qquad \text{for even } n$$

with $r \in \{0, 1, 2, \ldots, a_{n+2} - 1\}$, the spacing at $\boldsymbol{m}$ is

$$s(\boldsymbol{m}) = \sqrt{\frac{2U(\boldsymbol{m})}{\alpha}} \begin{cases} -(q_n + rq_{n+1})\alpha + (p_n + rp_{n+1}) & \text{for odd } n \\ (q_n + rq_{n+1})\alpha - (p_n + rp_{n+1}) & \text{for even } n. \end{cases}$$

Proof. Firstly consider the case where n is odd. First assume that the next lattice point passed through $\boldsymbol{m}'$ is right of $\boldsymbol{m}$, as perceived from the origin. Thus the coordinates x, y are positive and negative, respectively. Then we have the bound $|y| < p_{n+1}$. But from lemma 1.1

$$\begin{aligned} x\alpha + y &\geqslant (q_{n-1} + (q_{n+1} - 1)q_n)\alpha - (p_{n-1} + (a_{n+1} - 1)p_n) \\ &= (q_{n+1}\alpha - p_{n+1}) + (-q_n\alpha + p_n) \end{aligned}$$

whereas if the next lattice point $\boldsymbol{m}'$ is left of $\boldsymbol{m}$ as perceived from the origin, the coordinates x, y are negative and positive respectively, and

$$x\alpha + y \leqslant -q_n\alpha + p_n$$

which is a smaller spacing, thus the spacing corresponds to the latter. Then from lemma 1.1 we see that the minimum spacing such that $\boldsymbol{m}'$ lies in the positive quadrant is where

$$x\alpha + y = -(q_n + rq_{n+1})\alpha + (p_n + rp_{n+1}).$$

The case of n odd is analogous with

$$x\alpha + y = (q_n + rq_{n+1})\alpha - (p_n + rp_{n+1}).$$

Thus the theorem is proven. □

The regions described above fill out the quarter plane so the spacings at all points $\boldsymbol{m}$ are known. The rectangular regions shall be denoted as boxes $B_{n,r}$ where convenient. This is illustrated in figure 1, so for any lattice point that lies inside the box $B_{n,r}$ indicated (including the bottom and left side of the box only), the vector that determines the spacing is of the form $(x, y) = (-q_n - rq_{n+1}, p_n + rp_{n+1})$. This result has been derived in a different fashion by Pandey *et al* [8, 9]. There the values of $m_1\alpha + m_2$ over intervals $[M, M+1]$ are considered as $M \to \infty$.

The level-spacing distribution $P(s)$ is determined as follows. First evaluate the number of points $\boldsymbol{m}$ with spacing in the interval $(s, s+\Delta s)$ such that $\boldsymbol{m}$ is below the energy surface with area U, and divide this by the total number of points within the surface bounding U (which will be U itself as lattice points have density unity in $\boldsymbol{m}$ space). This will give a distribution $P(s, U)$ that estimates the level-spacing distribution $P(s)$. Then take the limit of this as $U \to \infty$ (if it exists). So $P(s, U)$ is defined by

$$\begin{aligned} \int_s^{s+\Delta s} P(s, U)\,\mathrm{d}s &= \frac{1}{U}\#\{\boldsymbol{m}|s(\boldsymbol{m}) \in (s, s+\Delta s), U(\boldsymbol{m}) \leqslant U\} \\ &= U^{-1} \sum_{\{\boldsymbol{m}|s(\boldsymbol{m})\in(s,s+\Delta s),U(\boldsymbol{m})\leqslant U\}} 1 \end{aligned} \qquad (10)$$

where # counts the number of points in the set, and

$$P(s) = \lim_{U\to\infty} P(s, U).$$

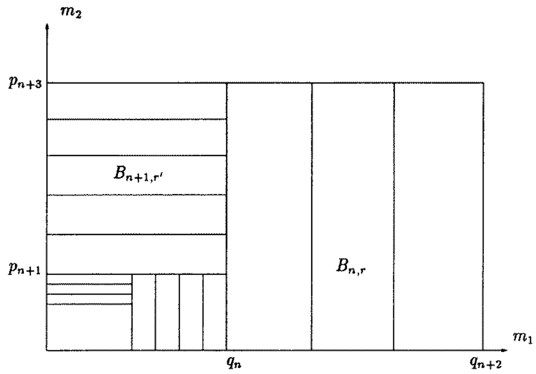

Figure 1. Spacings from the energy surface.

This limit does not exist for the following reasons, however. Consider U as it increases from $U_0 \simeq \frac{1}{2}\alpha q_n^2$ to $U_1 \simeq \frac{1}{2}\alpha q_{n+2}^2$, i.e. as the surface moves through boxes $\bigcup_{r=0}^{a_{n+2}-1} B_{n,r}$. $P(s, U_0)$ is then defined from $\frac{1}{2}\alpha q_n^2$ spacings, but by the time $U = U_1$, the spacings that make up $P(s, U_0)$ account for $\frac{1}{2}\alpha q_n^2$ out of $\frac{1}{2}\alpha q_{n+2}^2$ spacings for $P(s, U_1)$, thus they account for a proportion $(q_n/q_{n+2})^2$ of $P(s, U_1)$ which can be made as small as we like by taking large values for a_{n+1} or a_{n+2}. So $P(s, U)$ is largely dependent upon the boxes $B_{n,r}$ that the energy surface is passing through and the previous boxes soon lose impact upon $P(s, U)$ as U increases.

From [11] the behaviour of the sequence a_i is ergodic, implying that the behaviour of $P(s, U)$ is largely 'random', so no limit is to be expected generically for $U \to \infty$.

Also from [11] we have the result that for generic α

$$\lim_{n\to\infty} \frac{\ln q_n}{n} = \frac{\pi^2}{12\ln 2} \qquad \text{so } q_n^2 \simeq e^{n\pi^2/6\ln 2} \text{ for large } n, \tag{11}$$

suggesting that the box dimensions increase exponentially. Thus to obtain a convergent expression it would seem sensible to average over an exponential range of U. This is defined in the next section and shown to be convergent for all α.

3. The exponential average

Putting $U = e^W$, the exponential average is defined by

$$\overline{P(s)} = \lim_{V\to\infty} \frac{1}{V} \int_0^V P(s, e^W)\, dW$$

which on substitution of (10) reduces to

$$\int_s^{s+\Delta s} \overline{P(s)}\, ds = \lim_{V\to\infty} \frac{1}{V} \sum_{\{\boldsymbol{m} | s(\boldsymbol{m})\in(s,s+\Delta s), U(\boldsymbol{m})\leqslant e^V\}} U(\boldsymbol{m})^{-1}. \tag{12}$$

In what follows the sum over all points will be turned into a sum over the boxes $B_{n,r}$ for $n \leqslant N$ and the limit $V \to \infty$ converted to the limit $N \to \infty$. Then considering the contribution to $\int_s^{s+\Delta s} \overline{P(s)}\, ds$ from points inside the boxes $\bigcup_{n\leqslant N} \bigcup_{r=0}^{a_{N+2}-1} B_{N,r}$

$$\int_s^{s+\Delta s} \overline{P(s)}\, ds = \lim_{N\to\infty} \frac{1}{V} \left\{ \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \sum_{\{\boldsymbol{m} | s(\boldsymbol{m})\in(s,s+\Delta s), \boldsymbol{m}\in B_{n,r}\}} U(\boldsymbol{m})^{-1} + E(N) \right\}$$

where $2\ln(q_{N+2}+q_{N+1}) < V < 2\ln(q_{N+3}+q_{N+2})$ and $E(N)$ arises from points bounded by the energy surface and boxes $B_{n,r}$ with $n > N$, i.e. from the set $\{\boldsymbol{m}|\boldsymbol{m} \in B_{n,r},\ n > N$ and $U(\boldsymbol{m}) \leqslant \mathrm{e}^V\}$. Now there is the bound

$$E(N) < \sum_{\{\boldsymbol{m}|\frac{1}{2}\alpha q_{N+1}^2 < U(\boldsymbol{m}) < \frac{1}{2}\alpha(q_{N+2}+q_{N+3})^2\}} U(\boldsymbol{m})^{-1}$$

so considering the strip of points parallel to the constant energy surface that hit the m_1 axis along the interval $x, x+\Delta x$ where $x \in (q_{N+1}, q_{N+2}+q_{N+3})$ then these points will have the same U value for small Δx and the number of such points is the area (up to O(1)) ΔA. Then

$$\Delta A = \alpha x \Delta x \qquad \text{and} \qquad U^{-1} = \frac{2}{\alpha x^2}$$

so

$$E(N) < \int_{q_{N+1}}^{q_{N+2}+q_{N+3}} \frac{2}{x}\,\mathrm{d}x = 2\ln\left(\frac{q_{N+2}+q_{N+3}}{q_{N+1}}\right).$$

Then to estimate the total error, using the bounds on V

$$\frac{E(N)}{V} < \frac{\ln(q_{N+2}+q_{N+3}) - \ln(q_{N+1})}{\ln(q_{N+2}+q_{N+1})}$$

which will be seen to tend to zero as $N \to \infty$ in the examples that follow. Approximating V by $2\ln q_N$ will also have no effect. So

$$\int_s^{s+\Delta s} \overline{P(s)}\,\mathrm{d}s = \lim_{N\to\infty} \frac{1}{2\ln q_N} \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \sum_{\{\boldsymbol{m}|s(\boldsymbol{m})\in(s,s+\Delta s),\boldsymbol{m}\in B_{n,r}\}} U(\boldsymbol{m})^{-1}. \tag{13}$$

Now define

$$\int_s^{s+\Delta s} \overline{P_{n,r}(s)}\,\mathrm{d}s = \tfrac{1}{2} \sum_{\{\boldsymbol{m}|s(\boldsymbol{m})\in(s,s+\Delta s),\boldsymbol{m}\in B_{n,r}\}} U(\boldsymbol{m})^{-1} \tag{14}$$

so

$$\overline{P(s)} = \lim_{N\to\infty} \frac{1}{\ln q_N} \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \overline{P_{n,r}(s)}. \tag{15}$$

For a point $\boldsymbol{m}$ in the box $B_{n,r}$ the spacing is, by theorem 1

$$s(\boldsymbol{m}) = \sqrt{\frac{2U(\boldsymbol{m})}{\alpha}}|(q_n + rq_{n+1})\alpha - (p_n + rp_{n+1})|. \tag{16}$$

So inside any particular box as the energy surface sweeps through it with increasing U, $s(\boldsymbol{m})$ increases monotonically with it. Thus if there are any points with spacing in the interval $(s, s+\Delta s)$ they will lie in a strip parallel to the energy surface.

Let z be the place the strip first hits an axis (i.e. the axes m_1 and m_2 for n odd and even, respectively) with (small) variation Δz. The area of the strip, ΔA, will (at least asymptotically) be the number of points in $B_{n,r}$ with spacing inside $(s, s+\Delta s)$. For small Δz, all points will have the same U value of

$$U = \begin{cases} \dfrac{\alpha z^2}{2} & \text{for } n \text{ even} \\[2ex] \dfrac{z^2}{2\alpha} & \text{for } n \text{ odd.} \end{cases}$$

Also

$$\Delta A = \begin{cases} \alpha \Delta z(z - (q_n + rq_{n+1})) & q_n + rq_{n+1} \leqslant z \leqslant q_n + (r+1)q_{n+1} \\ \alpha \Delta z(q_n + (r+2)q_{n+1} - z) & q_n + (r+1)q_{n+1} \leqslant z \leqslant q_n + (r+2)q_{n+1} \end{cases}$$

for n even, and for n odd

$$\Delta A = \begin{cases} \dfrac{\Delta z}{\alpha}(z - (p_n + rp_{n+1})) & p_n + rp_{n+1} \leqslant z \leqslant p_n + (r+1)p_{n+1} \\ \dfrac{\Delta z}{\alpha}(p_n + (r+2)p_{n+1} - z) & p_n + (r+1)p_{n+1} \leqslant z \leqslant p_n + (r+2)p_{n+1}. \end{cases}$$

Note that substituting αz for z in the odd case converts it into the even, so for any n we have

$$U = \frac{\alpha z^2}{2}$$

and

$$\Delta A = \begin{cases} \alpha \Delta z(z - (q_n + rq_{n+1})) & q_n + rq_{n+1} \leqslant z \leqslant q_n + (r+1)q_{n+1} \\ \alpha \Delta z(q_n + (r+2)q_{n+1} - z) & q_n + (r+1)q_{n+1} \leqslant z \leqslant q_n + (r+2)q_{n+1}. \end{cases}$$

Now

$$\begin{aligned} & s(\boldsymbol{m}) = z|(q_n\alpha - p_n) + r(q_{n+1}\alpha - p_{n+1})| \\ \Rightarrow & \frac{\mathrm{d}s}{\mathrm{d}x} = |(q_n\alpha - p_n) + r(q_{n+1}\alpha - p_{n+1})| \end{aligned}$$

so by (13)

$$\int_s^{s+\Delta s} \overline{P_{n,r}(s)}\,\mathrm{d}s = \frac{1}{2}\frac{2}{\alpha z^2}\Delta A$$

yielding

$$\int_s^{s+\Delta s} \overline{P_{n,r}(s)}\,\mathrm{d}s = \begin{cases} \dfrac{\Delta z}{z^2}(z - (q_n + rq_{n+1})) & \text{if } q_n + rq_{n+1} \\ & \quad \leqslant z \leqslant q_n + (r+1)q_{n+1} \\ \dfrac{\Delta z}{z^2}(q_n + (r+2)q_{n+1} - z) & \text{if } q_n + (r+1)q_{n+1} \\ & \quad \leqslant z \leqslant q_n + (r+2)q_{n+1}. \end{cases}$$

Small Δz gives $\mathrm{d}s/\mathrm{d}z \simeq \Delta s/\Delta z$, so

$$\overline{P_{n,r}(s)} = \begin{cases} \dfrac{s - s_0}{s^2} & \text{if } s_0 \leqslant s \leqslant s_1 \\ \dfrac{s_2 - s}{s^2} & \text{if } s_1 \leqslant s \leqslant s_2 \end{cases}$$

where

$$s_i = (q_n + (r+i)q_{n+1})|(q_n\alpha - p_n) + r(q_{n+1}\alpha - p_{n+1})|.$$

Then using (8)

$$s_i = \frac{(x_n + r + i)(1 - ry_n)}{1 + x_n y_n} \qquad \text{where } x_n = \frac{q_n}{q_{n+1}}, \ \ y_n = \frac{1}{a'_{n+2}}.$$

Thus we have

Theorem 2. For α such that

$$\lim_{N\to\infty} \frac{\ln(q_{N+2}+q_{N+3}) - \ln q_{N+1}}{\ln(q_{N+2}+q_{N+1})} = 0 \tag{17}$$

the level-spacing distribution averaged over an exponential range of U converges to

$$\overline{P(s)} = \lim_{N\to\infty} \frac{1}{\ln q_N} \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \overline{P_{n,r}(s)} \tag{18}$$

provided the limiting sum exists, where

$$\overline{P_{n,r}(s)} = \begin{cases} \dfrac{s-s_0}{s^2} & \text{if } s_0 \leqslant s \leqslant s_1 \\ \dfrac{s_2-s}{s^2} & \text{if } s_1 \leqslant s \leqslant s_2 \end{cases}$$

and

$$s_i = \frac{(x_n+r+i)(1-ry_n)}{1+x_ny_n} \qquad \text{where } x_n = \frac{q_n}{q_{n+1}},\ \ y_n = \frac{1}{a'_{n+2}}.$$

4. Examples of the distribution

The following are examples of the average probability spacing distribution for various values of α. Berry and Tabor [5] give numerical results for values $1/\sqrt{2}$, $1/\sqrt{5}$, $1/\mathrm{e}$, $1/\pi$ upon which analytical observations shall be made. The simplest example, however, is the golden mean, which has the greatest level repulsion due to it having the slowest continued fraction convergence. Then (see Hardy and Wright [10]) $\alpha = [1, 1, 1, \ldots] = (1+\sqrt{5})/2$ which gives $x_n = (\sqrt{5}-1)/2 = y_n$ and $\ln q_N = N\ln((\sqrt{5}+1)/2)$. This means $r = 0$ and $\overline{P_{n,r}(s)}$ has no n dependency, thus

$$\overline{P(s)} = \begin{cases} \dfrac{s-1/\sqrt{5}}{s^2\ln((1+\sqrt{5})/2)} & \text{if } \dfrac{1}{\sqrt{5}} \leqslant s \leqslant \dfrac{3+\sqrt{5}}{2\sqrt{5}} \\ \dfrac{(2+\sqrt{5})/\sqrt{5}-s}{s^2\ln((1+\sqrt{5})/2)} & \text{if } \dfrac{3+\sqrt{5}}{2\sqrt{5}} \leqslant s \leqslant \dfrac{2+\sqrt{5}}{\sqrt{5}}. \end{cases}$$

The cases of $\alpha = 1/\sqrt{2}, 1/\sqrt{5}$, which both have periodic continued fraction expansions, give averaged distributions that are similar to those of [5]. For the case of $\alpha = \mathrm{e}^{-1}$, however, a significant difference exists. This has expansion $1/\mathrm{e} = [0, 2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, \ldots]$ (see [12]) which, satisfying the criterion of the following result, gives a delta function, whereas [5] reports a non-zero distribution.

Theorem 3. The averaged spacing distribution for $\alpha = [a_0, a_1, \ldots]$ is the delta function $\delta(s)$ provided (17) is satisfied and

$$\lim_{N\to\infty} \frac{N}{\ln(a_1 \ldots a_N)} = 0. \tag{19}$$

Proof. Note that if the a_i are of polynomial order with respect to i then (19) is satisfied as is (17), which is needed for convergence of the exponential average. By theorem 2

$$\overline{P(s)} = \lim_{N\to\infty} \frac{1}{\ln q_N} \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \overline{P_{n,r}(s)}.$$

Now define for any small ϵ

$$I(\epsilon) = \lim_{N\to\infty} \frac{1}{\ln q_n} \sum_{n\leqslant N} \sum_{r=0}^{a_{n+2}-1} \int_0^{\epsilon} \overline{P_{n,r}(s)}\, \mathrm{d}s$$

which if shown to be unity proves the theorem.

Note that the only contributions to the integral occur in the cases where $s_0 \leqslant \epsilon$. By considering possible values of r one can see that only the first interval of functions $\overline{P_{n,r}}$ can possibly contribute for small enough ϵ if $r = 0$. Hence

$$I(\epsilon) = \lim_{N\to\infty} \frac{1}{\ln q_N} \sum_{\{n\leqslant N | s_0 \leqslant \epsilon\}} \int_{s_0}^{\epsilon} \frac{s - s_0}{s^2}\, \mathrm{d}s$$

$$= \lim_{N\to\infty} \frac{N}{\ln q_N} \frac{1}{N} \sum_{\{n\leqslant N | x_n \leqslant \epsilon\}} \left[\ln\epsilon - \ln x_n + \frac{x_n}{\epsilon} - 1\right].$$

The main contributions arise from small values of x_n, i.e. the larger a_n values. So considering the asymptotics

$$q_N \simeq \prod_{i=1}^{N} a_i \qquad x_N \simeq \frac{1}{a_{N+1}}$$

then

$$I(\epsilon) = \lim_{N\to\infty} \frac{1}{\ln \prod_{i=1}^{N} a_i} [\ln\epsilon - 1] + \lim_{N\to\infty} \frac{1}{\ln \prod_{i=1}^{N} a_i} \sum_{\{n\leqslant N | 1/a_{n+1} \leqslant \epsilon\}} \frac{1}{\epsilon a_{n+1}}$$

$$+ \lim_{N\to\infty} \frac{\ln \prod_{\{n\leqslant N | a_{n+1} \leqslant 1/\epsilon\}} a_i}{\ln \prod_{i=1}^{N} a_{n+1}}$$

which with (19) gives

$$\lim_{N\to\infty} \frac{1}{\ln \prod_{i=1}^{N} a_i} [\ln\epsilon - 1] = 0$$

$$\lim_{N\to\infty} \frac{1}{\ln \prod_{i=1}^{N} a_i} \sum_{\{n\leqslant N | 1/a_{n+1} \leqslant \epsilon\}} \frac{1}{\epsilon a_{n+1}} \leqslant \lim_{N\to\infty} \frac{N}{\ln \prod_{i=1}^{N} a_i} = 0.$$

Now if

$$\lim_{N\to\infty} \frac{\ln \prod_{n\leqslant N | a_{n+1} \leqslant \epsilon} a_{n+1}}{\ln \prod_{i=1}^{N} a_i} = 1 \tag{20}$$

then as $I(\epsilon)$ is just a straight integral of a probability distribution, $I(\epsilon) \leqslant 1$, and for any $\epsilon > 0$, $I(\epsilon) = 1$, which gives the required result. Then it remains to see that (20) is true. From equation (19), for any (small) $k > 0 \exists m(k)$ such that $\forall N > m(k)$

$$\frac{N}{\ln \prod_{\{n\leqslant N\}} a_n} < k \iff \ln \prod_{\{n\leqslant N\}} a_n > \frac{N}{k}.$$

Now

$$\frac{\ln \prod_{\{n\leqslant N | a_{n+1} > 1/\epsilon\}} a_{n+1}}{\ln \prod_{\{n\leqslant N\}} a_n} = 1 - \frac{\ln \prod_{\{n\leqslant N | a_{n+1} < 1/\epsilon\}} a_{n+1}}{\ln \prod_{\{n\leqslant n\}} a_n}$$

so

$$0 < \frac{\ln \prod_{\{n\leqslant N | a_{n+1} < 1/\epsilon\}} a_{n+1}}{\ln \prod_{\{n\leqslant N\}} a_n} < \frac{k \ln \prod_{\{n\leqslant N | a_{n+1} < 1/\epsilon\}} a_{n+1}}{N} < \frac{k \ln(1/\epsilon)^N}{N} = k \ln(1/\epsilon).$$

Then as ϵ is fixed at this stage, k can be made arbitrarily small, and the result follows. □

It has since been proved that the Hausdorff dimension of the points satisfying this condition is $\frac{1}{2}$. This can be found in [13].

5. The generic case

The exponentially averaged spacing distribution has been seen to converge for various examples. However, to get a physically meaningful result the general frequency ratios need to be investigated. Then general x_n and y_n are being considered. This can be obtained via some ergodic analysis, where a generic distribution for $\overline{P(s)}$ will be obtained for generic α, i.e. for all frequency ratios except a set of Lebesgue measure zero. This distribution is shown in figure 2. The joint distribution of x_n and y_n will be advantageous. First define

$$T(x_n, y_n) = (x_{n+1}, y_{n+1})$$

then from (8) we find, for $[m]$ = integer part of m, $\{m\} = m - [m]$

$$T(x, y) = \left(\frac{1}{[1/y] + x}, \left\{\frac{1}{y}\right\}\right) \qquad T^{-1}(x, y) = \left(\left\{\frac{1}{x}\right\}, \frac{1}{[1/x] + y}\right). \tag{21}$$

Then an invariant density can be defined as

$$\rho(x, y) = \frac{1}{\ln 2} \frac{1}{(1 + xy)^2}$$

which can be shown by direct substitution to obey

$$\rho(x, y) = \rho(T(x, y)) = \rho(T^{-1}(x, y)).$$

Then define a measure over $[0, 1] \times [0, 1] \equiv I \times I$ as

$$\mu(A) = \int_A \frac{\mathrm{d}x\, \mathrm{d}y}{(1 + xy)^2}$$

for any subset A of $I \times I$. The mapping T can be seen to be ergodic with respect to the measure μ as follows. First define

$$x = [0, x_1, x_2, x_3, \ldots] \qquad y = [0, y_1, y_2, y_3, \ldots]$$

then if $T(x, y) = (T_1(x, y), T_2(x, y))$, from (21), for any k

$$T_1^k(x, y) = [0, y_k, y_{k-1}, y_{k-2}, \ldots, y_1, x_1, x_2, \ldots]$$
$$T_2^k(x, y) = [0, y_{k+1}, y_{k+2}, \ldots].$$

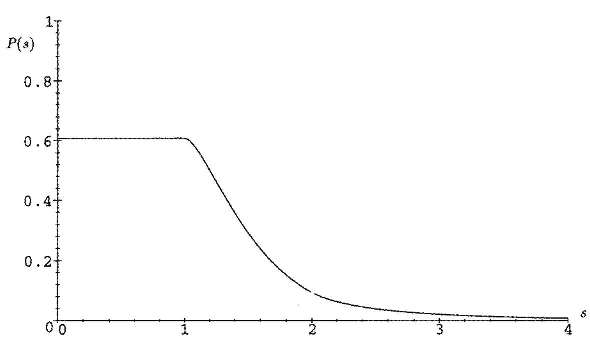

Figure 2. The generic averaged spacing distribution.

So using the invariance of the measure

$$\lim_{n\to\infty}\int_0^1\int_0^1 f(x,y)g(T^n(x,y))\,\mathrm{d}\mu = \lim_{n\to\infty}\int_0^1\int_0^1 f(T^k(x,y))g(T^{n+k}(x,y))\,\mathrm{d}\mu$$
$$= \lim_{n\to\infty}\int_0^1\int_0^1 f(T^k(x,y))g(T^{n+k}(x,y))\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}.$$

But since k can be arbitrarily large, $T_1^k(x,y)$ can be approximated arbitrarily closely by the function $\overline{T}_1^k(x,y)$ where

$$\overline{T}_1^k(x,y) = [0, y_k, y_{k-1}, \ldots, y_1, 0, 0, 0, \ldots].$$

Because $\overline{T}_1^k(x,y)$ is independent of x, the x dependence of $T_1^k(x,y)$ can be ignored. So integrate out x from the measure μ to obtain

$$\lim_{n\to\infty}\int_0^1 f(T^k(x,y))g(T^{n+k}(x,y))\frac{\mathrm{d}y}{1+y} \tag{22}$$

which is just the Gaussian measure. From [11], the Gauss map is mixing, thus (22) becomes

$$\int_0^1 f(T^k(x,y))\frac{\mathrm{d}y}{1+y}\int_0^1 g(T^k(x,y))\frac{\mathrm{d}y}{1+y}$$
$$= \int_0^1\int_0^1 f(T^k(x,y))\,\mathrm{d}\mu\int_0^1\int_0^1 g(T^k(x,y))\,\mathrm{d}\mu$$
$$= \int_0^1\int_0^1 f(x,y)\,\mathrm{d}\mu\int_0^1\int_0^1 g(x,y)\,\mathrm{d}\mu.$$

Then since this works for arbitrarily large k the required condition for mixing and hence ergodicity is obtained. Then from the Birkhoff–Kinchin ergodic theorem (see [11]) for suitably convergent functions f

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n<N} f(T^k(x,y)) = \frac{1}{\ln 2}\int_0^1\int_0^1 \frac{f(x,y)}{(1+xy)^2}\,\mathrm{d}x\,\mathrm{d}y. \tag{23}$$

Also by [11]

$$\lim_{N\to\infty}\frac{\ln q_N}{N} = \frac{\pi^2}{12\ln 2}. \tag{24}$$

Although this is proved in [11], equation (23) yields a much simpler proof, given in appendix B, which is an example of the power of (23). Now (24) means the error term of (17) vanishes, so combining (18) with these two results, and noting that $a_{n+2} = [1/y]$ (where the square brackets indicate the integer part), obtain

$$\overline{P(s)} = \frac{12}{\pi^2}\int_0^1\int_0^1\sum_{r=0}^{[1/y]-1}\overline{P_r(s,x,y)}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}$$

which rearranges to give

$$\overline{P(s)} = \frac{12}{\pi^2}\sum_{r=0}^{\infty}\int_0^{1/(r+1)}\int_0^1\overline{P_r(s,x,y)}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2}.$$

This is a direct integral, the evaluation of which is briefly outlined in appendix A. Thus we have

Theorem 4. The averaged spacing distribution of the spectrum of the quantized two-dimensional simple harmonic oscillator of generic frequency ratio is

$$\overline{P(s)} = \begin{cases} \dfrac{6}{\pi^2} & \text{if } 0 \leqslant s \leqslant 1 \\ \dfrac{6}{\pi^2 s} + \dfrac{12}{\pi^2}\left[\left(\dfrac{s-1}{s}\right)^2 \ln\left(\dfrac{s-1}{s}\right) - \dfrac{1}{4}\left(\dfrac{s-2}{s}\right)^2 \ln\left|\dfrac{s-2}{s}\right|\right] & \text{if } s \geqslant 1. \end{cases} \tag{25}$$

Various observations can be made about $\overline{P(s)}$ (see figure 2). Firstly, by direct calculation the total probability and mean, i.e. $\int_0^\infty \overline{P(s)}\,\mathrm{d}s$ and $\int_0^\infty s\overline{P(s)}\,\mathrm{d}s$, can be shown to be unity. This is not an automatic result of the distributions definition because, although $P(s, U)$ will have an average and total probability of one, in the limit of $U \to \infty$ this may not be the case. Take the situation of theorem 3, for example, where the total probability is unity but the average is zero. For large s we find from the expansion of (25)

$$\overline{P(s)} = \frac{4}{\pi^2 s^3} + \mathrm{O}(s^{-4}).$$

Now the mean of the distribution is unity, thus the variance of the distribution is found from

$$\mathrm{Var}(s) = \int_0^\infty (s-1)^2 \overline{P(s)}\,\mathrm{d}s.$$

However, the integral will then have a term of order $\mathrm{O}(s^{-1})$ to be integrated over $(1, \infty)$ which is divergent so the variance and hence the standard deviation is unbounded. The distribution can directly be seen to be a C^1 function. From figure 2 the reduced level clustering expected from the papers of Pandey *et al* [8, 9] can be seen.

6. Conclusions

It has been shown by Pandey *et al* [8, 9] that the level-spacing distribution for the two-dimensional harmonic oscillator is unstable. By defining a suitable average, a level-spacing distribution that converges under this semiclassical limit was obtained. This is unstable to perturbation of the ratio however, with a dense set of ratios having a delta function as the level-spacing distribution (by theorem 3). This set has Hausdorff dimension $\frac{1}{2}$, as shown in [13]. This differs significantly from the Poisson universality class associated with integrable systems.

Although the exponential average has been shown to converge, nothing has been stated about the magnitude of the oscillations of $P(s, U)$ with U. The variance of these oscillations can be defined, but does not yield easily to the type of analysis applied above.

This type of analysis cannot be extended easily to higher dimensions due to the lack of continued fraction theory that works so well in two. Using the Jacobi–Perron generalization (see [14]) is a possible further study. From the analysis for two dimensions, it is not clear that a stable distribution will necessarily be found in higher dimensions.

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Appendix A

Here the evaluation of

$$\overline{P(s)} = \frac{12}{\pi^2}\sum_{r=0}^{r=\infty}\int_0^{1/(r+1)}\int_0^1 \overline{P_r(s,x,y)}\frac{\mathrm{d}x\,\mathrm{d}y}{(1+xy)^2} \tag{A1}$$

is given, where

$$\overline{P_r(s,x,y)} = \begin{cases} \dfrac{s-s_0}{s^2} & \text{if } s_0 \leqslant s \leqslant s_1 \\ \dfrac{s_2-s}{s^2} & \text{if } s_1 \leqslant s \leqslant s_2 \end{cases} \tag{A2}$$

and

$$s_i = \frac{(x+r+i)(1-ry)}{1+xy}. \tag{A3}$$

Integrating with respect to y first, by defining

$$\overline{Q_r(s,x)} = \int_0^{1/(r+1)} \overline{P_r(s,x,y)}\frac{\mathrm{d}y}{(1+xy)^2} \tag{A4}$$

the following functions over the various intervals indicated are obtained. The functions are added whenever the intervals overlap:

$$\overline{Q_r(s,x)} = \frac{1}{2s^2} - \frac{1}{s(x+r)} + \frac{1}{2(x+r)^2} \qquad \frac{r+x}{r+x+1} \leqslant s \leqslant r+x$$

$$\overline{Q_r(s,x)} = -\frac{1}{2s^2} + \frac{1}{s(x+r+1)} + \frac{1}{2s^2(x+r+1)^2} \qquad \frac{r+x}{r+x+1} \leqslant s \leqslant 1$$

$$\overline{Q_r(s,x)} = -\frac{1}{s^2} - \frac{(s-1)^2}{s^2}\frac{1}{x+r} + \frac{1}{x+r+1} \qquad 1 \leqslant s \leqslant x+r+1$$

$$\overline{Q_r(s,x)} = \frac{1}{2s^2} + \frac{1}{2s^2(x+r+1)^2} + \frac{1-s}{s^2}\frac{1}{x+r+1} \qquad 1 \leqslant s \leqslant \frac{r+x+2}{r+x+1}$$

$$\overline{Q_r(s,x)} = \frac{1}{2s^2} + \frac{(s-2)^2}{4s^2} - \frac{1}{4}\frac{1}{x+r+2} \qquad \frac{x+r+2}{x+r+1} \leqslant s \leqslant x+r+2.$$

Thus y has been integrated out and the x integral remains. From equations (A1) and (A4)

$$\overline{P(s)} = \frac{12}{\pi^2}\sum_{r=0}^{\infty}\int_0^1 \overline{Q_r(s,x)}\,\mathrm{d}x. \tag{A5}$$

So defining

$$\overline{R_r(s)} = \int_0^1 \overline{Q_r(s,x)}\,\mathrm{d}x \tag{A6}$$

will just leave the summation to be calculated. Then $\overline{R_r(s)}$ will be calculated for various intervals of s, again the functions being added where the intervals overlap. Then

$$R_r(s) = a(s) = \frac{1}{2(1-s)} + \frac{1}{2} + \frac{1}{s}\ln(1-s) \qquad 0 \leqslant s \leqslant \frac{1}{2}$$

$$R_r(s) = b(s) = \frac{1}{2s^2} - \frac{1}{2} + \frac{1}{s}\ln s \qquad \frac{1}{2} \leqslant s \leqslant 1$$

$$R_r(s) = c_r(s) = \frac{1}{2s(1-s)} - \frac{r}{2s^2} - \frac{1}{s}\ln\left(\frac{s}{r(1-s)}\right) - \frac{1-s}{2s} + \frac{1}{2r}$$
$$\frac{r}{r+1} \leqslant s \leqslant \frac{r+1}{r+2} \qquad r \geqslant 1$$
$$R_r(s) = d_r(s) = \frac{1}{2s^2} - \frac{1}{s}\ln\left(\frac{r+1}{r}\right) - \frac{1}{2(1+r)} + \frac{1}{2r} \qquad \frac{r+1}{r+2} \leqslant s \leqslant r$$
$$R_r(s) = e_r(s) = \frac{1}{2s^2}(1+r-s) - \frac{1}{s}\ln\left(\frac{r+1}{s}\right) - \frac{1}{2(1+r)} + \frac{1}{2s} \qquad r \leqslant s \leqslant r+1$$
$$R_r(s) = f_r(s) = \frac{r}{2s^2} - \frac{1}{2s(1-s)} + \frac{1}{s}\ln\left(\frac{1}{(r+1)(1-s)}\right) - \frac{1-s}{2s^2} + \frac{1}{2s^2(r+1)}$$
$$\frac{r}{r+1} \leqslant s \leqslant \frac{r+1}{r+2} \qquad r \geqslant 0$$
$$R_r(s) = g_r(s) = -\frac{1}{2s^2} + \frac{1}{s}\ln\left(\frac{r+2}{r+1}\right) - \frac{1}{2s^2(r+2)} + \frac{1}{2s^2(r+1)} \qquad \frac{r+1}{r+2} \leqslant s \leqslant 1$$
$$R_r(s) = h_r(s) = -\frac{1}{s^2} - \left(\frac{s-1}{s}\right)^2 \ln\left(\frac{r+1}{r}\right) + \ln\left(\frac{r+2}{r+1}\right) \qquad 1 \leqslant s \leqslant r+1$$
$$R_r(s) = i_r(s) = \frac{s-r-2}{s^2} - \left(\frac{s-1}{s}\right)^2 \ln\left(\frac{r+1}{s-1}\right) + \ln\left(\frac{r+2}{s}\right) \qquad r+1 \leqslant s \leqslant r+1$$
$$R_r(s) = j_r(s) = \frac{1}{2s^2} - \frac{1}{2s^2(r+2)} + \frac{1}{2s^2(r+1)} + \frac{1-s}{s^2}\ln\left(\frac{r+2}{r+1}\right) \qquad 1 \leqslant s \leqslant \frac{r+3}{r+2}$$
$$R_r(s) = k_r(s) = \frac{1}{2s^2(s-1)} - \frac{r+1}{2s^2} - \frac{s-1}{2s^2} + \frac{1}{2s^2(r+1)} + \frac{1-s}{s^2}\ln\left(\frac{1}{(s-1)(r+1)}\right)$$
$$\frac{r+3}{r+2} \leqslant s \leqslant \frac{r+2}{r+1}$$
$$R_r(s) = l_r(s) = \frac{1}{2s^2} - \frac{1}{2s^2(s-1)} + \frac{r+1}{2s^2} + \frac{(s-2)^2}{4s^2}\ln\left(\frac{(r+1)(s-1)}{2-s}\right)$$
$$-\frac{1}{4}\ln\left(\frac{(r+3)(s-1)}{s}\right) \qquad \frac{r+3}{r+2} \leqslant s \leqslant \frac{r+2}{r+1}$$
$$R_r(s) = m_r(s) = \frac{1}{2s^2} + \frac{(s-2)^2}{4s^2}\ln\left(\frac{r+1}{r}\right) - \frac{1}{4}\ln\left(\frac{r+3}{r+2}\right) \qquad \frac{r+2}{r+1} \leqslant s \leqslant r+2$$
$$R_r(s) = n_r(s) = \frac{3+r-s}{2s^2} + \frac{(s-2)^2}{4s^2}\ln\left(\frac{r+1}{s-2}\right) - \frac{1}{4}\ln\left(\frac{r+3}{s}\right)$$
$$r+2 \leqslant s \leqslant r+3.$$

Now

$$\overline{P(s)} = \frac{12}{\pi^2}\sum_{r=0}^{\infty}\overline{R_r(s)} \tag{A7}$$

so to find $\overline{P(s)}$ just sum up $\overline{R_r(s)}$ over the various intervals listed above. First consider the interval $0 \leqslant s \leqslant \frac{1}{2}$, where

$$\overline{P(s)} = \frac{12}{\pi^2}[a(s) + f_0(s)] = \frac{6}{\pi^2}.$$

Now for the interval $\frac{1}{2} \leqslant s \leqslant \frac{2}{3}$,

$$\overline{P(s)} = \frac{12}{\pi^2}[b(s) + c_1(s) + f_1(s) + g_0(s)] = \frac{6}{\pi^2}.$$

Consider the intervals $\frac{r}{r+1} \leqslant s \leqslant \frac{r+1}{r+2}$ for $r \geqslant 2$ to give

$$\overline{P(s)} = \frac{12}{\pi^2}\left[b(s) + c_r(s) + \sum_{v=1}^{r-1} d_v(s) + f_r(s) + \sum_{v=0}^{r-1} g_v(s)\right]$$

which yields the same value on substitution. Thus $\overline{P(s)} = \frac{6}{\pi^2}$ for $0 \leqslant s \leqslant 1$.

Now consider $\frac{r+3}{r+2} \leqslant s \leqslant \frac{r+2}{r+1}$ for $r \geqslant 1$, where

$$\overline{P(s)} = \frac{12}{\pi^2}\left[\sum_{v=2}^{\infty} d_v(s) + e_1(s) + \sum_{v=1}^{\infty} h_v(s) + i_0(s)\sum_{v=0}^{r-1} j_v(s) + k_r(s) + l_r(s) + \sum_{v=r+1}^{\infty} m_v(s)\right]$$
$$\iff \overline{P(s)} = \frac{12}{\pi^2}\left[\frac{1}{2s} + \left(\frac{1}{s} - \frac{3}{4}\right)\ln s + \frac{(s-1)^2}{s^2}\ln(s-1) - \frac{(s-2)^2}{4s^2}\ln(2-s)\right]$$

which is independent of r, so true for the range $1 \leqslant s \leqslant \frac{3}{2}$. For $\frac{1}{2} \leqslant s \leqslant 2$

$$\overline{P(s)} = \frac{12}{\pi^2}\left[\sum_{v=2}^{\infty} d_v(s) + e_1(s) + \sum_{v=1}^{\infty} h_v(s) + i_0(s) + k_0(s) + l_0(s) + \sum_{v=1}^{\infty} m_v(s)\right]$$
$$\iff \overline{P(s)} = \frac{12}{\pi^2}\left[\frac{1}{2s} + \left(\frac{1}{s} - \frac{3}{4}\right)\ln s + \frac{(s-1)^2}{s^2}\ln(s-1) - \frac{(s-2)^2}{4s^2}\ln(2-s)\right]$$

which is now for the range $1 \leqslant s \leqslant 2$. Finally consider ranges for $r \geqslant 2$ of $r \leqslant s \leqslant r+1$ to give

$$\overline{P(s)}\frac{12}{\pi^2}\left[\sum_{v=r+1}^{\infty} d_v(s) + e_r(s) + \sum_{v=r}^{\infty} h_v(s) + i_{r-1}(s) + \sum_{v=r-1}^{\infty} m_v(s) + n_{r-2}(s)\right]$$
$$\iff \overline{P(s)} = \frac{12}{\pi^2}\left[\frac{1}{2s} + \left(\frac{1}{s} - \frac{3}{4}\right)\ln s + \frac{(s-1)^2}{s^2}\ln(s-1) - \frac{(s-2)^2}{4s^2}\ln(s-2)\right]$$

which altogether give the function outlined above, i.e.

$$\overline{P(s)} = \begin{cases} \dfrac{6}{\pi^2} & \text{if } 0 \leqslant s \leqslant 1 \\ \dfrac{6}{\pi^2 s} + \dfrac{12}{\pi^2}\left[\left(\dfrac{s-1}{s}\right)^2 \ln\left(\dfrac{s-1}{s}\right) - \dfrac{1}{4}\left(\dfrac{s-2}{s}\right)^2 \ln\left|\dfrac{s-2}{s}\right|\right] & \text{if } s \geqslant 1. \end{cases}$$

□

Appendix B

As has already been remarked, in [11], the following asymptotic formula is proved:

$$\ln q_n = \frac{\pi^2 N}{12\ln 2}.$$

Using equation (23), the same result can be obtained as follows:

$$\ln q_n = \ln\left(\frac{q_n}{q_{n_1}}\frac{q_{n-1}}{q_{n-2}}\cdots\frac{q_2}{1}\right) = -\sum_{\{n<N\}} \ln x_n.$$

Then by (23)

$$\ln q_n \simeq \frac{N}{\ln 2}\int_0^1\int_0^1 \frac{\ln x}{(1+xy)^2}\,\mathrm{d}y\,\mathrm{d}x = -\frac{N}{\ln 2}\int_0^1 \frac{\ln x}{1+x}\,\mathrm{d}x = \frac{\pi^2 N}{12\ln 2}.$$

This a lot simpler than the proof in [11], and a good example of the power of (23).

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